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About theoretical and practical impact of mesh adaptation on approximation of functions and PDE solutions

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SUMMARY

We try to formalize and study how mesh adaptation improves the approximation of interpolated functions or of PDE solutions. We first define an adaptive solution, in the sense that the pair (mesh,function) satisfies a non-linear coupled equation. In order to build optimal mesh adaptation strategies, we also define a functional model, the 'continuous metric', which leads to propose the best mesh for a given function and a given norm. We then describe how convergence of adaptive solutions can be better than for non-adaptive ones; this involves some recent refinements concerning what we called early capturing of details, a specific property of good adaptive strategies. We give some typical numerical illustrations. Convergence properties depend very much on how mesh adaptation is performed and we exhibit theoretical limits for the maximum order of accuracy reachable for some family of mesh adaptation methods. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: mesh adaptation; fluid mechanics; interpolation

1. INTRODUCTION

Very soon, CFD engineers and scientists have been faced to non-smooth solutions of Partial Differential Equations (PDEs) for which meshes should be adapted. During the last 20 years, mesh adaptation was the centre of many works, which we can classify in three groups, according to the mesh technique used. If the mesh is deformed, with a constant topology, then it was soon understood that the mesh also is a solution of some system, an important idea for the sequel. If the mesh is locally divided, researchers have looked for better error estimates, and in particular to *a posteriori* error estimates. See Reference [1] for a pioneering work or [2, 3] for more recent developments. If the mesh is more strongly modified or even

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rebuilt, then the question of adequate stretching was risen and well advanced for example in References [4–6]. While *a posteriori* error estimates are undoubtedly the major domain of progress in mesh adaptation, they do not answer alone some important questions about mesh adaptation. In the proposed study, we restrict ourselves to a context for which error estimation is trivial. We consider the interpolation of a given function on a mesh. This will allow to focus on complementary questions and in particular:

(a) How can one adapt a mesh to the best interpolation of a given function?
(b) What are the convergence properties of an adapted interpolation?
(c) What is this practically useful for?

In order to answer question (a), we suggest not to look directly for a class of meshes. Instead, we propose to modellize the properties of a class of meshes starting from the idea of a metric, assumed to be a continuous function. In each point of the computational domain, the metric defines the local size and stretching of an ideal mesh. Since it is a function, we can apply a functional treatment of the research of an optimal mesh.

Concerning question (b), it is generally believed that mesh adapted methods can be approximations of high order for singular functions. We recall that mesh adaptation is also essential for smooth functions with large gradients. Then we explain how performances of some family of mesh adaptation methods are limited by some barriers that we shall describe.

About question (c), we like to show that mesh adaptation methods are not (not only?) an extra complication in the process of solving PDE. On the contrary, mesh adaptation provides discrete solutions that converge faster and easier to the continuous limit, while, in many computations, usual strategies of refinement definitively fail in showing good 'mesh convergence'. Once in the sweet paradise of mesh convergence, we can apply old recipes à la Romberg, yielding some idea of the local error size and possibly still better solutions.

This leads us to the following plan:

2. Adaptive solution;
3. Continuous metric;
4. Adaptive convergence to singular functions;
5. Early capturing;
6. Barriers for refinement strategies;
7. Concluding remarks.

2. ADAPTIVE SOLUTION

A numerical scheme provides the approximate solution $u_{\mathscr{M}}$ of a given PDE when the mesh $\mathscr{M}$ is given.

Conversely we shall say that we have a *perfect mesh adaptor* if, given a PDE and its approximate solution, and given a number of nodes N, we can derive a unique mesh $\mathscr{M}$ adapted to the solution.

A perfect adaptive solution is a pair $(u, \mathscr{M})$ which is the fixed point of the above two steps: (a) an approximate solution u is computed on the mesh $\mathscr{M}$ and (b) a new mesh $\mathscr{M}$ is derived by mesh adaptation from u.

We refer to [5, 6] for works presenting this kind of iteration.

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3. CONTINUOUS METRIC

A set of unstructured meshes is not easily mapped into a vector space, due to the heterogeneity of the data used for defining the topology of each mesh. Furthermore, two *equivalent* meshes, i.e. that produce two numerical solutions of same quality, are not represented by the same amount of information if their topologies differ. It is then interesting to represent a family of equivalent meshes by a unique function defined on the computational domain, that specifies the local fineness and stretching that is common to equivalent meshes.

3.1. Definitions

We restrict here to the one-dimensional case. A metric on interval $[a,b]$ is a continuous positive-valued function $\mathscr{M}$ of the spatial independent variable $x \in [a,b]$.

We shall say that a mesh of N points : $x_1 = a, x_2, \ldots, x_N = b$ belongs to the equivalence class related to metric $\mathscr{M}$ if the difference:

$$\int_{x_i}^{x_{i+1}} \sqrt{\mathscr{M}}\, \mathrm{d}x - 1$$

is much smaller than 1 for any i. Assume for simplicity that it is zero, the *local mesh size* $m_{\mathscr{M}} = \mathscr{M}^{-1/2}$ satisfies:

For any interval $[x_i, x_{i+1}]$ of a mesh specified by metric $\mathscr{M}$, *the following equality holds*: $\int_{x_i}^{x_{i+1}} 1/m_{\mathscr{M}}\, \mathrm{d}x = 1$.

Conversely, given a metric $\mathscr{M}$, we can directly compute its *number of nodes* $C(\mathscr{M})$:

$$C(\mathscr{M}) = \int_0^1 (\mathscr{M})^{1/2}\, \mathrm{d}x = \int_0^1 1/m_{\mathscr{M}}\, \mathrm{d}x \tag{1}$$

Let u be a given twice differentiable function. We denote by $\Pi_{\mathscr{M}}$ the continuous, linear by element, interpolator on mesh $\mathscr{M}$. The interpolation error can be *modelled* as follows (cf. Reference [7]):

$$|e_{\mathscr{M}}(x)| = |(u - \Pi_{\mathscr{M}} u)(s)| = m_{\mathscr{M}}^2 |u''(x)| \tag{2}$$

where m plays the role of Taylor's formula step Δx.

3.2. Optimal metric

Given a function u, instead of looking for the best mesh, we are now able to look for the best metric, by minimizing in the L^α norm of the $\mathscr{P}_1$ interpolation error induced by the metric; this leads to a minimum problem with respect to the metric:

$$\text{Find } \mathscr{M}_N = \operatorname{Arg}\min_{\mathscr{M}} (|e_{\mathscr{M}}(x))_{L^\alpha}| \tag{3}$$

under the constraint:

$$\int_0^1 m_{\mathscr{M}}^{-1}(x)\, \mathrm{d}x = N \tag{4}$$

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Using the model of error, we get

$$\frac{1}{2}\int_0^1 |e_{\mathcal{M}}(x)|^\alpha \,\mathrm{d}s = \frac{1}{2}\int_0^1 \left(m_{\mathcal{M}}^2 \left|\frac{\partial^2 u}{\partial x^2}\right|\right)^\alpha \mathrm{d}s \tag{5}$$

We discard index $\mathcal{M}$ for simplicity. The usual calculus of variations applied to $1/m$ (to have a linear constraint) yields the following optimality condition:

$$(2\alpha)\int_0^1 (1/m)^{-2\alpha-1}(|u''|)^\alpha \delta(1/m)\,\mathrm{d}s \geqslant 0 \quad \forall\delta(1/m): \int_0^1 \delta(1/m) = 0 \tag{6}$$

and the optimal metric $\mathcal{M}_N$ is given by its mesh size:

$$m_N(x) = \frac{\int |u''|^{\alpha/(2\alpha+1)}\,\mathrm{d}s}{N}\,|u''(x)|^{-\alpha/(2\alpha+1)} \tag{7}$$

The above formula specifies the node distribution when the second derivative of u and the total number N of nodes are given.

This method extends well to the *multi-dimensional case* and *anisotropic meshes*, see Reference [7].

3.3. Towards the PDE case

This theory also extends to the research of an optimal mesh for an *approximate PDE solution*. We shall give here only a few hints about the way it does.

Let us assume we are solving the Dirichlet problem:

$$-u_{xx} = f \quad \textit{on}\ [0,1]; u(0) = u(1) = 0 \tag{8}$$

We can modellize the approximation error from the *a priori* estimate as

$$\frac{1}{2}\int_0^1 |e_{\mathcal{M}}(x)|^\alpha \,\mathrm{d}s = \frac{1}{2}\int_0^1 (-\Delta)^{-1}\left(m^2\left|\frac{\partial^2 u}{\partial x^2}\right| + m^2\left|\frac{\partial^2 f}{\partial x^2}\right|\right)^\alpha \mathrm{d}s \tag{9}$$

where $(-\Delta)^{-1}g$ stands for the solution of the Dirichlet problem with right-hand side g. The minimization of functional (9) under state equation (8) is an Optimal Control problem for which optimality is obtained by introducing the classical adjoint state [8].

4. ADAPTIVE CONVERGENCE TO SINGULAR FUNCTIONS

4.1. Higher-order convergence

Let us assume that we have computed a set of adaptive solutions (according to definition of Section 1) $(\mathcal{M}_N, U_N)$ with any N of a sequence of positive integers tending to infinity. Then the mesh convergence order α can be expressed in terms of the total number N of nodes and of the dimension of geometrical space d (here $d=1$):

$$\text{error} \leqslant Cte\, N^{\alpha/d}$$

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In the case of a discontinuous function u, sequences of uniform meshes cannot show an order better than one-half. A higher-order convergence property of adaptive strategies is asserted by the following lemma:

Lemma 1
Let us consider an interpolator of real-valued functions of one variable, with theoretical order of accuracy order α. Let u a piecewise regular function with one discontinuity. Then uniform refinement convergence can be of order at most $\frac{1}{2}$ in L^2. At the contrary, when an adaptive strategy is applied, the convergence order can be as high as α.

Indeed, in short, an adaptive strategy can force the successive meshes to be $2^{(\alpha+1)}$ finer near the discontinuity while just twice finer in regular parts. Then the global error is 2^α times smaller, while the number of nodes is essentially doubled. Then the method is of order α.

4.2. Convergence order of the continuous metric model

Let us examine how to look for an optimal metric in the *discontinuous case*. When a $\mathscr{P}_1$ interpolation is choosen, we modellize the error as

$$\int_0^1 |e_{\mathscr{M}}(x)|^\alpha \,\mathrm{d}s = \int_0^1 (m^2|\delta^{-2}(u(x+\delta) - 2u(x) + u(x-\delta))|)^\alpha \,\mathrm{d}s \tag{10}$$

Where δ is smaller than m. We observe that the differential quotient:

$$\delta^{-2}(u(x+\delta) - 2u(x) + u(x-\delta)):$$

- is close to $\partial^2 u/\partial x^2$ where u is regular;
- or of the order of δ^{-2} at singularities of u.

Moreover, since u is bounded,

$$||\delta^{-2}(u(x+\delta) - 2u(x) + u(x-\delta))||_{L^{1/2}} \textit{ is bounded independently of } \delta \tag{11}$$

The calculus of variations gives now:

$$m_N(x) = Cte.|(|\delta^{-2}(u(x+\delta) - 2u(x) + u(x-\delta))|(x))|^{-2/5} \tag{12}$$

and the resulting optimal error in L^2 writes:

$$\text{Error} = \frac{2}{N^2}\left(\int |\delta^{-2}(u(x+\delta) - 2u(x) + u(x-\delta))|^{2/5}\right)^{5/2} < \frac{K}{N^2}$$

where K is a bounded constant, due to (11). We deduce that the adaptive strategy is formally of second-order accuracy.

Lemma 1 in previous section and the analysis of present section extend to multidimensional cases. There is still a lot of work in order to study how these properties can be generalized to the context of mesh adaptation applied to PDEs. The next section tries to show some numerical evidence that the above theory applies well to CFD.

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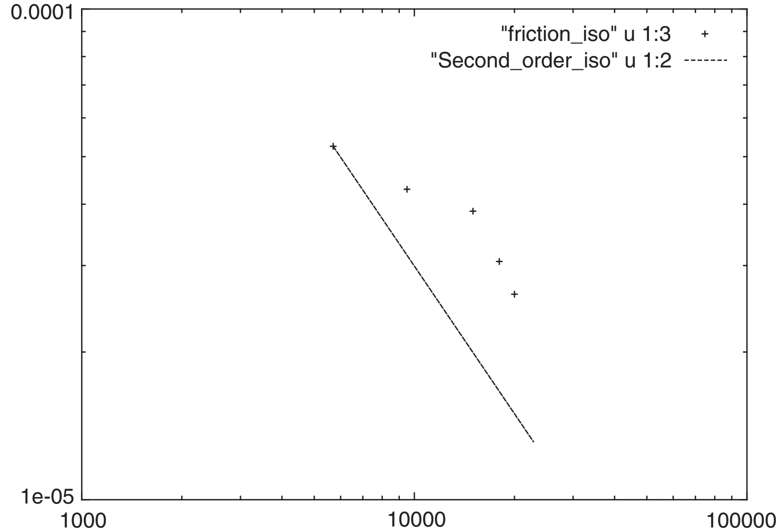

Figure 1. Mesh convergence of friction velocity for a flat plate computation with isotropic adaptation. Number of nodes is read on x-axis, L^2 approximation error on y-axis. Crosses correspond to a series of adapted isotropic computations, dash line for second-order convergence. Second-order convergence is observed only for mesh sizes larger than 10 000.

5. EARLY CAPTURING

Many authors have noted that in the case of smooth functions with locally high gradients, uniform refinement may show half-order convergence until many nodes are considered in such a way that local mesh size is small enough for the good representation of the local high gradients. Only then, the higher-order convergence appears. See for example Reference [9]. In contrast, the number of nodes necessary for adaptive methods to show higher-order convergence is much smaller number. We call this property 'early capturing'. (Figs. 1 and 2).

5.1. Early capturing in CFD

In order to illustrate this property, we apply a mesh adaptation strategy that is a rather popular combination of Hessian evaluation and anisotropic mesh reconstruction with control on the mesh size [4, 5]. Iteration between PDE solution and mesh adaptation is applied for a fixed number N of nodes.

We study the capture of a boundary layer in a supersonic turbulent flow past a flat plate. The flow model is a k–ε one with a wall law parameterized in such a way that only the fully turbulent part of the boundary layer is captured by the mesh. We consider the friction velocity on a point on the plate after the stagnation point. A very fine and accurate adapted anisotropic computation with 200 000 nodes plays the role of the reference exact solution.

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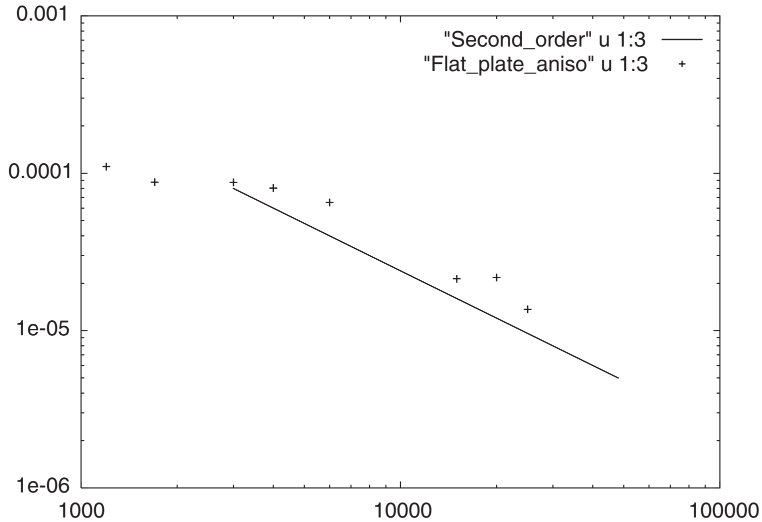

Figure 2. Mesh convergence of friction velocity for a flat plate computation with anisotropic adaptation. Number of nodes is read on x-axis, L^2 approximation error on y-axis. Crosses correspond to a series of adapted anisotropic computations, dash line for second-order convergence. Second-order convergence is already observed for mesh sizes larger than 3000.

A series of uniform refinements with meshes as large as 100 000 nodes failed to produce a measurable convergence to solution. We compare then two adaptive strategies: an isotropic one in which triangles are not too different from equilateral and an anisotropic one in which stretching is applied. We observe that the isotropic strategy allows early capturing, i.e. second-order convergence with a rather small number of nodes, i.e. for meshes finer than 15 000 nodes.

The anisotropic strategy give second-order convergence already for 6000 nodes. Other examples are presented in Reference [10].

5.2. Certified convergence in CFD

In the previous section, we get a mesh convergence at the asymptotical order. We examine now how we can derive from it an evaluation of the accuracy of the result, and therefore a kind of 'certification' of its quality.

In the experiments presented in the previous section, we compare our calculations with a reference 'very accurate' solution, but in practice it is not available. Instead, we can measure the numerical order of convergence by comparing three results obtained with meshes involving N nodes, $4N$ nodes and $16N$ nodes.

As a test case, we consider the very easy computation of a laminar flow around a NACA0012 airfoil. Farfield Mach number is 1.2 and Reynolds number is 73. We start with uniform refinement with a coarse mesh of rather good quality, with 3000 nodes (= vertices). It is uniformly divided first into a 12 000-node mesh which, in turn, is divided uniformly in a 48 000-node mesh. We then compute the numerical order of convergence. Table I shows the results. This gives only a first-order numerical convergence.

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Table I. Numerical convergence order in L^2 norm for an airfoil flow.

Flow variables	Error for (a) uniform (fine)	Error for (b) adaptive (coarse)	Error for (c) adaptive (fine)
Density, ρ	0.948	1.62	2.15
Horizontal moment, ρU	1.016	1.78	1.75
Vertical moment, ρV	1.024	1.85	1.87

Note: Mesh sequence (a) is obtained by uniform refinement, with meshes of 3000, 12 000, 48 000 nodes, mesh sequence (b) is obtained with coarser anisotropic adaptated meshes of 800, 3000, 12 000 nodes, mesh sequence (c) is obtained with anisotropic adaptated meshes of 3000, 12 000, 48 000 nodes.

Conversely, if each successive mesh is built with anisotropic adaptation, a convergence order of 1.6–1.8 is already observed with a coarser sequence of 800–3000–12 000-node meshes. Second-order convergence is confirmed by a finer sequence of adapted meshes with the two previous adapted meshes, 3000–12 000-node, complemented by a 48 000-node adapted mesh.

Extrapolation strategies can then be applied to specific outputs (lift, drag, etc.), see for example Reference [11].

Let us denote by u_N the solution computed with the adapted anisotropic mesh of N vertices. From the computation we get

$$||u_{3000} - u_{12\,000}||_{L^2} = 1.802 \times 10^{-4}$$

Relying on the second-order convergence we can *certify* the order of magnitude of the error on the medium mesh as follows:

$$||u_{12\,000} - u_{\text{exact}}||_{L^2} \leqslant (1.802/3)10^{-4} = 0.600 \times 10^{-4} \tag{13}$$

As expectable, this estimate is coherent with the finest calculation which gives

$$||u_{12000} - u_{48\,000}||_{L^2} = 5.637 \times 10^{-5}$$

6. BARRIERS FOR REFINEMENT STRATEGIES

The results of previous section point out difference of performances between isotropic and anisotropic mesh adaptation. In some cases, we can quantify this difference, thanks to the statement of a theoretical accuracy upper limit or barrier for isotropic mesh adaptation methods:

Lemma 2 (*Coudière* [12])
The order of convergence α in L^p of an isotropic adaptive mesh method applied in dimension d to $\mathscr{P}_1$-interpolate a discontinuous function is bounded by

$$\alpha \leqslant \frac{d/p}{d-1}$$

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Then isotropic mesh refinement is a rather inefficient procedure for interpolating discontinuous functions: the error estimate with order $\frac{3}{4}$ for isotropic 3D adaptation is a severe limit. Many engineers and scientists remarked that the price of isotropic mesh adaptation such as Adaptive Mesh refinement is reasonable in 2D, but prohibitive in 3D, as soon as large 2D surfaces of discontinuity are involved. Our conjecture—and numerical experience—is that the above limit does not apply to best anisotropic mesh adaptation methods.

7. CONCLUDING REMARKS

The above remarks, although not yet proposing a coherent and complete analysis, tend to show that mesh adaptation should not be considered only as a mesh improvement technique, but rather as a new approximation method.

The adapted mesh is a part of the solution, it tends to be uniquely defined. The system to solve is strongly non-linear. Iterations between mesh and flow are mandatory. We propose a way to realize this program by building first a complete functional context relying on the continuous metric. The system appears then as the optimality condition of an optimization problem. We are currently studying the extension of these results to multidimensional PDEs.

A specific convergence theory for adaptive meshes can be built by focusing on singular functions. Adaptive methods then do converge with a better order of accuracy that non-adaptive ones.

But isotropic adaptive methods have limitations in accuracy order for discontinuous functions. The order is much smaller than for smooth functions. Anisotropic adaptive methods likely escape these limitations.

In practical cases of aerodynamics, these limitations apply to most available mesh sizes. Conversely, the proposed mesh adaptive algorithms performed well on pre-industrial flow cases.

From this point of view, mesh adaptation is a key for safer numerical computations, i.e. computations with a good control of accuracy.

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